Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Approximation Theory 138 (2006) 168–183
Journal of Approximation Theory
www.elsevier.com/locate/jat

Hypercyclic sequences of PDE-preserving operators☆

Henrik Petersson

School of Mathematical Sciences, Chalmers/Göteborg University, SE-412 96 Göteborg, Sweden

Received 24 June 2005; accepted 7 November 2005
Available online 4 January 2006

Abstract

A sequence $\mathbb{T} = (T_n)$ of continuous linear operators $T_n : \mathscr{X} \to \mathscr{X}$ is said to be hypercyclic if there exists a vector $x \in \mathscr{X}$, called hypercyclic for $\mathbb{T}$, such that $\{T_n x : n \geqslant 0\}$ is dense. A continuous linear operator, acting on some suitable function space, is PDE-preserving for a given set of convolution operators, when it map every kernel set for these operators invariantly. We establish hypercyclic sequences of PDE-preserving operators on $\mathscr{H}(\mathbb{C}^d)$, and study closed infinite-dimensional subspaces of, except for zero, hypercyclic vectors for these sequences.

MSC: Primary 47A16; 47B38; 32A15; secondary 47A15; 32W99

Keywords: Hypercyclic sequence; Universal; Hypercyclic subspace; Hypercyclic spectrum; PDE-preserving; Entire function

1. Introduction and notation

In what follows, $\mathcal{X}$ denotes an arbitrary separated real or complex locally convex space. The algebra of continuous linear operators on $\mathcal{X}$ is denoted by $\mathcal{L}(\mathcal{X})$. By $\mathbb{N}$ we denote the set of non-negative integers. A sequence $\mathbb{T} = (T_n)_{n \geqslant 0}$ of operators $T_n \in \mathcal{L}(\mathcal{X})$ is said to be *hypercyclic* (or universal) if there exists a vector $x \in \mathcal{X}$, called hypercyclic for $\mathbb{T}$, such that $\{T_n x : n \in \mathbb{N}\}$ is dense. A single operator $T \in \mathcal{L}(\mathcal{X})$ is said to be hypercyclic when the sequence (T^n) of powers is and, accordingly, the hypercyclic vectors for T are those for (T^n). The importance of hypercyclicity derives from the study of invariant subsets; for example, $T \in \mathcal{L}(\mathcal{X})$ lacks non-trivial closed invariant subsets if and only if every vector $x \neq 0$ is hypercyclic for T. A *hypercyclic subspace* for $\mathbb{T}$ (T), is a closed infinite-dimensional subspace $H \subseteq \mathcal{X}$ whose non-zero vectors are hypercyclic vectors for $\mathbb{T}$ (T). The study of hypercyclic subspaces has become of great interest for different

☆ Supported by The Royal Swedish Academy of Sciences.
E-mail address: henripet@math.chalmers.se.

0021-9045/$ - see front matter
doi:10.1016/j.jat.2005.11.004

reasons. For example, it is known that every hypercyclic operator supports a dense invariant subspace of, except for zero, hypercyclic vectors, see [9,14]; in contrast, there are hypercyclic operators that do not admit any hypercyclic subspace, see e.g. [24, Theorem 3.4]. We refer to [7,10,13,15,17,24,30,31] for work on this latter topic, which we shall discuss more below. We also refer to [18] for an excellent and exhaustive survey of the theory of hypercyclicity.

In this note we study hypercyclic sequences of operators acting on the Fréchet space $\mathcal{H} = \mathcal{H}(\mathbb{C}^d)$ of entire functions in d variables equipped with the compact-open topology (uniform convergence on compact sets). Hypercyclicity in this setting has been studied in for example [2,4,7,16,29]. In particular, we recall the well-known result of Godefroy and Shapiro [16, Theorem 5.2], saying that every non-trivial convolution operator on $\mathcal{H}$ is hypercyclic.

Definition 1. A convolution operator on $\mathcal{H}$ is an operator $T \in \mathcal{L}(\mathcal{H})$ that commutes with all translations $\tau_a : f \mapsto f(a+z)$. (A scalar multiple of the identity operator is called a trivial convolution operator.) The set of convolution operators on $\mathcal{H}$ is denoted by $\mathcal{C}$.

It follows that convolution operators commute, i.e., $\mathcal{C}$ forms a commutative subalgebra of $\mathcal{L}(\mathcal{H})$. In fact, we have the following nice description of $\mathcal{C}$ (see [16, Section 5] for details). Let $\mathrm{Exp} = \mathrm{Exp}(\mathbb{C}^d)$ denote the algebra of exponential type functions, i.e., the set of all $\varphi \in \mathcal{H}$ such that $|\varphi(z)| \leqslant M e^{r\|z\|}$ for some $M, r > 0$, where $\|z\| \equiv \sqrt{\sum_i |z_i|^2}$ if $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$. Then we have:

Proposition 1. *The map* $\varphi = \sum_{\alpha \in \mathbb{N}^d} \varphi_\alpha z^\alpha \mapsto \varphi(D) \equiv \sum_\alpha \varphi_\alpha D^\alpha$ *defines an algebra isomorphism between* Exp *and* $\mathcal{C}$. *Here* $D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}$, $D_i \equiv \partial/\partial z_i$, *and* $\sum_\alpha \varphi_\alpha D^\alpha f$ *converges in* $\mathcal{H}$ *for all* $f \in \mathcal{H}$.

Thus, in particular, if p is a polynomial, $p(D)$ is the differential operator obtained by replacing each variable z_i in p by D_i.

In [4] (see also [8]) Bernal–González extended the result of Godefroy and Shapiro to sequences $(\varphi_n(D))$ of convolution operators. He obtained the following result:

Proposition 2 (*Bernal–González*). *Let* $(\varphi_n(D))$ *be a sequence in* $\mathcal{C}$. *Then* $(\varphi_n(D))$ *is hypercyclic provided* (P) *or* (Q) *holds where*:

(P) *There are non-empty open subsets* $U, V \subseteq \mathbb{C}^d$ *such that for any pair of finite subsets* E, F *of* U *and* V, *respectively, there is a subsequence* (n_k) *of positive integers such that* $\varphi_{n_k}(x) \to 0$ *and* $\varphi_{n_k}(y) \to \infty$ *for all* $x \in E$ *and* $y \in F$.

(Q) $\min\{|\alpha| : D^\alpha \varphi_n(0) \neq 0\} \to \infty$ $(n \to \infty, |\alpha| \equiv \sum \alpha_i)$ *and there is a non-empty open subset* $U \subseteq \mathbb{C}^d$ *such that for any finite subset* $E \subseteq U$, *there is a subsequence* (n_k) *of positive integers such that* $\varphi_{n_k}(x) \to \infty$ *for all* $x \in E$.

In particular (P) extends Godefroy and Shapiro's Theorem, since if $\varphi_n = \varphi^n$ and $\varphi \in \mathrm{Exp}$ is not constant, we may take $U = \{z : |\varphi(z)| < 1\}$ and $V = \{z : |\varphi(z)| > 1\}$.

We shall extend and complement the work of Godefroy, Shapiro and Bernal–González in the following way. We study hypercyclic sequences of operators from a larger class than $\mathcal{C}$, namely, from the class of so-called PDE-preserving operators.

Definition 2. An operator $T \in \mathcal{L}(\mathcal{H})$ is PDE-preserving for a given set $\mathbb{E} \subseteq \mathrm{Exp}$ if it maps every kernel set $\ker \varphi(D)$, $\varphi \in \mathbb{E}$, invariantly, that is, $\varphi(D)Tf = 0$ whenever $\varphi(D)f = 0$ $(f \in \mathcal{H})$. The set of PDE-preserving operators for $\mathbb{E}$ is denoted by $\mathcal{O}(\mathbb{E})$.

Note that, for any set $\mathbb{E}$, $\mathcal{O}(\mathbb{E})$ forms a subalgebra of $\mathcal{L}(\mathcal{H})$ and, in turn, $\mathcal{C}$ is a commutative subalgebra of $\mathcal{O}(\mathbb{E})$. PDE-preserving operators have been studied in [26–28]. In particular, by $\mathbb{H}$ we denote the set of homogeneous polynomials, and we have the following characterization result for $\mathcal{O}(\mathbb{H})$. Let $\mathcal{S}$ denote the set of sequences $\Phi = (\varphi_n)_{n \geqslant 0}$ in Exp such that, for some $N, M, r > 0$, $|\varphi_n(z)| \leqslant N M^n e^{r\|z\|}$ for all n and z, see also Proposition 7 for an alternative description of $\mathcal{S}$. By H_n we denote the projector on $\mathcal{H}$ onto the space $\mathcal{H}_n$ of n-homogeneous polynomials defined by $f = \sum_{m \geqslant 0} f_m \mapsto f_n$, where $\sum f_m$ is the Taylor expansion of $f \in \mathcal{H}$ about the origin. We have now (see e.g. [27, Theorem A]):

Proposition 3. *The algebra $\mathcal{O}(\mathbb{H})$ of* PDE*-preserving operators for $\mathbb{H}$ is formed by the operators $\Phi(D) \equiv \sum_{n \geqslant 0} H_n \varphi_n(D)$, $\Phi = (\varphi_n) \in \mathcal{S}$. The sequence $\Phi \in \mathcal{S}$ is unique for $\Phi(D)$, i.e., $\mathcal{S} \simeq \mathcal{O}(\mathbb{H})$.*

Note that for $\varphi(D) \in \mathcal{C}$, $\varphi(D) = \Phi(D)$ where Φ is the constant sequence $(\varphi_n = \varphi)$. In other words, $\mathcal{C}$ corresponds in $\mathcal{S}$ to the set of constant sequences. An important example of a non-convolution operator in $\mathcal{O}(\mathbb{H})$ is the Euler operator $\langle z, D\rangle \equiv z_1 D_1 + \cdots + z_d D_d$. For any $m \geqslant 1$ we have that $\langle z, D\rangle^m = \Phi(D)$ where Φ is the sequence $(\varphi_n \equiv n^m)_{n \geqslant 0} \in \mathcal{S}$ of constant mappings.

More specifically the objective in this note is to study sequences $(\Phi_n = (\varphi_{n,m})_m)_n$ in $\mathcal{S}$ such that the corresponding sequence $(\Phi_n(D))$ in $\mathcal{O}(\mathbb{H})$ is hypercyclic. (In [29] we studied hypercyclic *operators* in $\mathcal{O}(\mathbb{H})$, some of the results there follow from the more general results obtained here.) For our purpose we shall apply the following well-known Hypercyclicity Criterion (HC), that originates from Carol Kitai's thesis, see [11] for a proof:

Proposition 4 (*Hypercyclicity criterion*). *Let $\mathbb{T} = (T_n)$ be a sequence of operators $T_n \in \mathcal{L}(\mathcal{F})$ on a separable Fréchet space $\mathcal{F}$. Assume $\mathbb{T}$ satisfies the HC for some sequence $(n_k) \subseteq \mathbb{N}$ in the sense that*: *There are dense subsets $X, Y \subseteq \mathcal{F}$* (*not necessarily linear*) *and a sequence of* (*possibly discontinuous*) *maps $\mathbb{S} = (S_{n_k} : Y \to \mathcal{F})$ such that*;

C1. *$T_{n_k} x \to 0$ as $k \to \infty$ for all $x \in X$,*
C2. *$S_{n_k} y \to 0$ as $k \to \infty$ for all $y \in Y$,*
C3. *$T_{n_k} S_{n_k} y \to y$ as $k \to \infty$ for all $y \in Y$.*

Then $\mathbb{T}$ is hereditarily hypercyclic *for the sequence (n_k), i.e., $(T_{m_k})_k$ is hypercyclic* (*in fact, the set* $\mathrm{hc}((T_{m_k}))$ *of hypercyclic vectors for $(T_{m_k})_k$ is dense in $\mathcal{F}$*) *for every subsequence (m_k) of (n_k).*

We say that $\mathbb{T}$ is hereditarily hypercyclic (satisfies the HC resp.) when $\mathbb{T}$ is hereditarily hypercyclic (satisfies the HC resp.) for some sequence in $\mathbb{N}$. Of course, hereditarily hypercyclic implies hypercyclic, and it has been studied intensively to what degree the converse holds, see [11] for more on this.

Another purpose is to study the existence of hypercyclic subspaces for the obtained classes of hypercyclic sequences. We apply the following criterion of Bonet et al. [13] (similar criterions can be found in [6,17,30]):

Proposition 5 (*Bonet, Martínez-Giménez, Peris*). *Let $\mathbb{T} = (T_n)$ be a sequence of operators $T_n \in \mathcal{L}(\mathcal{F})$ on a separable Fréchet space $\mathcal{F}$ that admits a continuous norm. Assume there exist a closed infinite-dimensional subspace $E \subseteq \mathcal{F}$ and a sequence (n_k), for which $\mathbb{T}$ satisfies the HC, such that $T_{n_k} \to 0$ pointwise on E. Then $\mathbb{T}$ has a hypercyclic subspace.*

(Note that $\mathcal{H}$ admits a continuous norm so the proposition applies.) Their proof rests on the remarkable fact, see [13, Theorem 3.1] and the proof of [13, Theorem 3.5], that if $\mathbb{T}$ satisfies the HC, then the corresponding sequence $L_{\mathbb{T}} \equiv (L_{T_n})$ in $\mathcal{L}(\mathcal{F})$ of left-multipliers $L_{T_n} : S \mapsto T_n S$ has a compact hypercyclic vector $K \in \mathcal{L}(\mathcal{F})$, for the strong operator topology (SOT). Recall that the SOT is the topology of pointwise convergence. Any such operator K maps non-zero vectors onto hypercyclic vectors for $\mathbb{T}$ (see [15, Proposition 7]). As a complement to this we prove (Theorem 5) that if a sequence $\mathbb{T} \subseteq \mathcal{L}(\mathcal{F})$ has a hypercyclic subspace H, then we can construct a compact one-to-one operator $K \in \mathcal{L}(\mathcal{F})$ with $\operatorname{Im} K \subseteq H$. This is one of the factors that motivate us to study the set of hypercyclic subspaces for our sequences of operators. In particular, we shall apply the following complement to Proposition 5, which we have obtained in a parallel work [31]:

Proposition 6. *Let $\mathbb{T} = (T_n)$ be a sequence of operators $T_n \in \mathcal{L}(\mathcal{F})$ on a separable Fréchet space $\mathcal{F}$ that admits a continuous norm. Assume there exist a complemented infinite-dimensional subspace $E \subseteq \mathcal{F}$ and a sequence (n_k), for which $\mathbb{T}$ satisfies the HC, such that $T_{n_k} \to 0$ pointwise on E. Then $\mathbb{T}$ has a complemented hypercyclic subspace H that is isomorphic to E.*

Recall that a subspace $E \subseteq \mathcal{X}$ is said to be complemented when there exists a subspace F such that $\mathcal{X} = E \oplus F$ and the corresponding projector $\mathcal{X} \to E$ (or equivalently, the projector $\mathcal{X} \to F$) is continuous. Every complemented subspace is closed and, by the closed-graph theorem, a subspace $E \subseteq \mathcal{F}$ of a Fréchet space $\mathcal{F}$ is complemented if and only if E is closed and $\mathcal{F} = E \oplus F$ for some closed F. (The existence of a complemented hypercyclic subspace H ensures thus that there exists a projector Π onto, except for zero, hypercyclic vectors.)

The paper is organized as follows. In Section 2 we obtain a sufficient condition (Theorem 1) for sequences $(\Phi_n(D))$ in $\mathcal{O}(\mathbb{H})$ to be hypercyclic. We apply this result to some special type of sequences. In particular, we complement Bernal–González' result by considering sequences in $\mathcal{C}$, i.e. by applying Theorem 1 in the case $\Phi_n(D) = \varphi_n(D) \in \mathcal{C}$, see Corollary 4. Another important application is to operators in $\mathcal{O}(\mathbb{H})$, i.e. when the sequence elements $\Phi_n(D)$ are of the form $\Phi_n(D) = \Phi(D)^n$, see Corollary 1. In Section 3 we obtain sufficient conditions for the obtained hypercyclic sequences in Section 2 to admit hypercyclic subspaces. In Section 4 we study the corresponding sets of hypercyclic subspaces, in particular we establish the existence of complemented hypercyclic subspaces. Finally, in the last Section 5, we discuss some consequences of our result based on the fact that hypercyclic properties of a sequence $\mathbb{T}$ impose that $L_{\mathbb{T}}$ is hypercyclic. In particular, it follows that for each obtained hypercyclic sequence $(\Phi_n(D))$, there exists an entire function $h = h(z, \xi) \in \mathcal{H}(\mathbb{C}^d \times \mathbb{C}^d)$ such that $h(\cdot, \xi)$ is hypercyclic for $(\Phi_n(D))$ for any fixed $\xi \in \mathbb{C}^d$, see Corollary 6.

2. Hypercyclic sequences

We recall that $\mathcal{H}_n$ denotes the vector space of n-homogeneous polynomials (in d variables). With standard multi-index notation, $\alpha! \equiv \prod \alpha_i!$, $|\alpha| \equiv \sum \alpha_i$ etc., we define an inner-product $(\cdot, \cdot)_n$ on $\mathcal{H}_n$ by $(p, q)_n \equiv \sum_{|\alpha|=n} p^{(\alpha)}(0)\overline{q^{(\alpha)}(0)}/\alpha!$ The symbol $\|\cdot\|_n$ denotes the corresponding norm defined by $\|p\|_n^2 \equiv (p, p)_n$. (See in (2) below how $\|\cdot\|_n$ relates to the sup-norm.)

The compact-open topology on $\mathcal{H} = \mathcal{H}(\mathbb{C}^d)$ is generated by the seminorms $\|f\|_{H:r} \equiv \sum_{n \geqslant 0} r^n \|H_n f\|_n/\sqrt{n!}$, $r > 0$ (this follows by inequalities (2) below). Let us also recall that H_n denotes the projector on $\mathcal{H}$ onto $\mathcal{H}_n$ defined by $f = \sum f_m \mapsto f_n$, where $\sum_{m \geqslant 0} f_m$ is the Taylor expansion of f about the origin.

The following estimates will play a central role:

$$\|p\|_n\|q\|_m \leqslant \|pq\|_{n+m} \leqslant 2^{n+m}\|p\|_n\|q\|_m, \tag{1}$$

where $p \in \mathcal{H}_n$ and $q \in \mathcal{H}_m$. If $p \in \mathcal{H}_n$, $p(D)$ maps $\mathcal{H}_{m+n}$ into $\mathcal{H}_m$, and the operator "multiplication by p", which we simply denote by p, maps $\mathcal{H}_m$ into $\mathcal{H}_{m+n}$. In fact, if $\bar{p}$ denotes the n-homogeneous polynomial obtained from p by conjugating its coefficients, $\bar{p} : \mathcal{H}_m \to \mathcal{H}_{m+n}$ is the Hilbert adjoint of $p(D) : \mathcal{H}_{m+n} \to \mathcal{H}_m$ which gives:

Lemma 1. *Let p be a non-zero element of $\mathcal{H}_n$. Then the composition $p(D)\bar{p}$ is an isomorphism on $\mathcal{H}$, and maps every $\mathcal{H}_m$ bijectively.*

Proof. We only prove the last statement, as this is actually what we need. Since $p(D)^* = \bar{p}$, $\ker p(D)^\perp = \operatorname{Im} \bar{p}$ and hence $\mathcal{H}_{m+n} = \ker p(D) \oplus \operatorname{Im} \bar{p}$. Next, $\bar{p}$ is injective so $p(D)^* = \bar{p}$ implies $p(D) : \mathcal{H}_{m+n} \to \mathcal{H}_m$ is surjective, and the statement follows. □

The first result, that $p(D)\bar{p}$ maps $\mathcal{H}$ isomorphically, is due to Shapiro [32], and the second statement is Ernst Fischer's classical theorem from 1911, see [32] for remarks.

Lemma 2. *Let p be a non-zero element of $\mathcal{H}_n$. Then $\bar{p}(p(D)\bar{p})^{-1}$ maps $\mathcal{H}_m$ into $\mathcal{H}_{m+n}$ with norm $\leqslant 1/\|p\|_n$.*

Proof. Put $P \equiv \bar{p}(p(D)\bar{p})^{-1}$ and let $f \in \mathcal{H}_m$. Then with $g \equiv (p(D)\bar{p})^{-1}f$, $\bar{p}g = Pf$ and Cauchy–Schwartz' inequality and (1) give:

$$\begin{aligned}\|f\|_m\|g\|_m &= \|p(D)\bar{p}g\|_m\|g\|_m \geqslant (p(D)\bar{p}g, g)_m = (\bar{p}g, \bar{p}g)_{m+n} = \|\bar{p}g\|_{m+n}^2\\ &\geqslant \|p\|_n\|Pf\|_{m+n}\|g\|_m,\end{aligned}$$

since $\|\bar{p}\|_m = \|p\|_m$. This proves the lemma. □

In view of our purposes, it is convenient to make use of the following alternative description of the sequence class $\mathcal{S}$ that we introduced in the paragraph before Proposition 3.

Proposition 7. *A sequence $\Phi = (\varphi_n)$ in* Exp *belongs to $\mathcal{S}$ if and only if there are constants $M, R, r > 0$, such that $\|H_m\varphi_n\|_m \leqslant MR^n r^m/\sqrt{m!}$ for all $m, n \geqslant 0$.*

Proof. By Cauchy's estimates, it is easily checked that $(\varphi_n) \in \mathcal{S}$ if and only if $\sup_{\|z\|\leqslant 1}|H_m\varphi_n(z)| \leqslant MR^n r^m/m!$ for some $M, R, r > 0$. Hence the proposition follows by the following relation between $\|\cdot\|_n$ and the sup-norm

$$\sup_{\|z\|\leqslant 1}|p(z)| \leqslant \|p\|_n/\sqrt{n!} \leqslant (n+1)^{d/2}d^{n/2}\sup_{\|z\|\leqslant 1}|p(z)|, \quad p \in \mathcal{H}_n \tag{2}$$

(see [32, p. 519], and see also the proof of Lemma 3 in [29], for further comments). □

Hence, in particular, a sequence (p_n) of homogeneous polynomials p_n belongs to $\mathcal{S}$ if and only if $\|p_n\|_{d(n)} \leqslant MR^n r^{d(n)}/\sqrt{d(n)!}$ for some $M, R, r > 0$, where $d(n)$ denotes the degree of p_n. So if $\Phi_n = (p_{n,m})_m$, $n \in \mathbb{N}$, are sequences of homogeneous polynomials and $d(n, m) \equiv \deg p_{n,m}$, then $(\Phi_n = (p_{n,m})_m)_n$ forms a sequence in $\mathcal{S}$ if and only if for each $n \geqslant 0$ there are constants

$M_n, R_n, r_n > 0$ such that

$$\|p_{n,m}\|_{d(n,m)} \leqslant M_n R_n^m r_n^{d(n,m)}/\sqrt{d(n,m)!}, \quad m = 0, 1, \ldots .$$

Theorem 1. *Let $(\Phi_n = (p_{n,m})_m)_n$ be a sequence in $\mathcal{S}$ where $p_{n,m}$ are homogeneous polynomials. Assume there is a sequence $(n_k) \subseteq \mathbb{N}$ such that $\lim_k d(k, m) = \infty$ for every fixed m, the sequence $(d(k, m))_{m \geqslant 0}$ is increasing for fixed k, and*

$$\lim_{k\to\infty} \inf_{m\geqslant 0} \sqrt{d(k,m)}\|p_{n_k,m}\|_{d(k,m)}^{1/d(k,m)} = \infty, \tag{3}$$

where $d(k, m) \equiv \deg p_{n_k,m}$. Then $(\Phi_n(D))$ satisfies the HC, and is thus hereditarily hypercyclic, for (n_k).

Proof. We prove that $(\Phi_n(D))$ satisfies the HC for (n_k) with X and Y being the set $\mathcal{P}$ of polynomials. We define $S_n : \mathcal{P} \to \mathcal{H}$ by $S_n \equiv \sum_{m\geqslant 0} \bar{p}_{n,m}(p_{n,m}(D)\bar{p}_{n,m})^{-1} H_m$. Using that $p(D)H_{m+n} = H_m p(D)$ for any n-homogeneous polynomial p, it is easily checked that S_n is a right inverse to $T_n \equiv \Phi_n(D)$ on $\mathcal{P}$, so C3 holds. We must prove that $S_{n_k} f \to 0$ for any $f \in \mathcal{P}$. By Lemma 2 we obtain for every $r > 0$ the estimate

$$\begin{aligned}
\|S_{n_k} f\|_{H:r} &= \sum_{m\geqslant 0} \frac{r^{m+d(k,m)}}{\sqrt{(m+d(k,m))!}} \|\bar{p}_{n_k,m}(p_{n_k,m}(D)\bar{p}_{n_k,m})^{-1} H_m f\|_{m+d(k,m)} \\
&\leqslant \sum_{m\geqslant 0} \frac{r^{d(k,m)}}{\sqrt{d(k,m)!}\,\|p_{n_k,m}\|_{d(k,m)}} \frac{r^m}{\sqrt{m!}} \|H_m f\|_m \\
&\leqslant \|f\|_{H:r} \sup_{m\geqslant 0} \frac{r^{d(k,m)}}{\sqrt{d(k,m)!}\|p_{n_k,m}\|_{d(k,m)}},
\end{aligned} \tag{4}$$

since $m!n! \leqslant (m+n)!$ and $m + d(k, m) \neq m' + d(k, m')$ if $m \neq m'$. Thus, we must prove that the last factor (the supremum-factor) in (4) tends to zero for every $r > 0$, and we claim that this is equivalent to (3). Indeed, by Stirling's formula, $(n!)^{1/n} \sim n/e$ $(n \to \infty)$, we may find constants $a, b > 0$ so that $an \leqslant (n!)^{1/n} \leqslant bn$ for all $n \geqslant 1$. Hence

$$(A\sqrt{d(k,m)})^{d(k,m)} \leqslant \sqrt{d(k,m)!} \leqslant (B\sqrt{d(k,m)})^{d(k,m)}$$

for all $k, m \geqslant 0$ for some $A, B > 0$, and our claim follows now easily (the details are left to the reader). Thus $S_{n_k} f \to 0$ so C2 holds. Finally, since $\lim_k d(k, m) = \infty$ for fixed m, $T_{n_k} f = 0$ for all k sufficiently large when f is a polynomial. Hence $T_{n_k} \to 0$ pointwise on $\mathcal{P} = X$, and C1–C3 have been established. □

Example 1. Let p be any non-zero k-homogeneous polynomial where $k \geqslant 1$, and put $p_{n,m} \equiv (m + kn)^n p^n$ and $\Phi_n \equiv (p_{n,m})_m$. Then $\Phi_n(D) = \sum_{m\geqslant 0} H_m p_{n,m}(D)$ equals $p(D)^n \langle z, D\rangle^n$. We note that $\deg p_{n,m} = kn$, and

$$\inf_{m\geqslant 0} \sqrt{kn}\|p_{n,m}\|_{kn}^{1/(kn)} = \sqrt{kn}(kn)^{1/k}\|p^n\|_{kn}^{1/(kn)} \geqslant \sqrt{kn}(kn)^{1/k}\|p\|_k^{1/k} \to \infty$$

as $n \to \infty$, since $\|p^n\|_{kn} \geqslant \|p\|_k^n$ by (1). Thus (Φ_n) satisfies the hypothesis of Theorem 1 for the full sequence (n), and so $(\Phi_n(D))$ is hypercyclic. In the same way (take $p_{n,m} \equiv (m + k)^n p$) we conclude that the sequence $(p(D)\langle z, D\rangle^n)$ is hypercyclic. (Parenthetically we note that

$(\langle z, D\rangle^n p(D)^n)$ is not hypercyclic, since $\langle z, D\rangle^n p(D)^n f$ vanishes at the origin for any (non-constant) $f \in \mathcal{H}$, so $1 \notin \overline{\{\langle z, D\rangle^n p(D)^n f\}_n}$. The same argument shows that $(\langle z, D\rangle^n p(D))$ fails to be hypercyclic.)

We apply Theorem 1 to some special constructions. First of all it gives us a short proof of the following result from [29]:

Corollary 1 (*Operators*). *Let* $\Phi = (p_n)$ *be a sequence of k-homogeneous polynomials, where* $k \geqslant 1$, *such that for some* $c, M, R > 0$, $c \leqslant \|p_n\|_k \leqslant MR^n$ *for all* $n \geqslant 0$. *Then* $\Phi \in \mathcal{S}$ *and* $\Phi(D) \in \mathcal{O}(\mathbb{H})$ *is (hereditarily) hypercyclic.*

Proof. Since $\mathcal{O}(\mathbb{H})$ is a ring, we have that $\Phi(D)^n = \sum H_m p_{n,m}(D)$ for some elements $p_{n,m} \in$ Exp. If $\Xi = (\xi_n)$, $\Psi = (\psi_n) \in \mathcal{S}$ and $\Xi(D)\Psi(D) = \sum H_n \varphi_n(D)$, then

$$\varphi_n = \sum_{m \geqslant 0} \xi_{n+m} H_m(\psi_n). \tag{5}$$

From this formula for composition in $\mathcal{O}(\mathbb{H})$, which is proved in [26] (Theorem 6), we obtain inductively that

$$p_{n,m} = p_m p_{m+k} p_{m+2k} \cdots p_{m+(n-1)k}.$$

Hence, $p_{n,m}$ are homogeneous polynomials where $d(n, m) = \deg p_{n,m} = kn$ and, by (1), $\|p_{n,m}\|_{d(n,m)} \geqslant c^n$. From this it is clear that the hypothesis of Theorem 1 is satisfied for the full sequence (n), hence $(\Phi(D)^n)$, i.e. $\Phi(D)$, is hereditarily hypercyclic. □

Example 2. Let p be a non-zero k-homogenous polynomial ($k \geqslant 1$) and $\Phi \equiv (p_n \equiv (n+k)p)_n$, then $\Phi(D) = p(D)\langle z, D\rangle$ and Corollary 1 gives that $\Phi(D)$ is hypercyclic.

Corollary 2 (*Shifts*). *Let* $\Phi = (p_n)$ *be a sequence of non-zero homogeneous polynomials* p_n *of degree* $k(n)$, *respectively, where the sequence* $(k(n))_n$ *is increasing and unbounded. Assume that the sequence* $(\sqrt{k(n)}\|p_n\|_{k(n)}^{1/k(n)} R^{-n/k(n)})_{n \geqslant 0}$ *is bounded for some* $R > 0$ *and that* $\lim_n \sqrt{k(n)}\|p_n\|_{k(n)}^{1/k(n)} = \infty$. *Then* $(\Phi_n(D))$, $\Phi_n \equiv (p_{n+m})_m$, *forms a (hereditarily) hypercyclic sequence in* $\mathcal{O}(\mathbb{H})$.

Proof. The hypothesis that $\sup_n \sqrt{k(n)}\|p_n\|_{k(n)}^{1/k(n)} R^{-n/k(n)}$ is finite implies $(p_{n,m} \equiv p_{n+m})_m \in \mathcal{S}$ for all n. We have that $d(n, m) = \deg p_{n,m} = k(n+m)$, so by the hypothesis on $k(n)$, it remains only to prove that (3) holds (for the full sequence). So let $\omega > 0$ be arbitrary and choose $N = N(\omega)$ so that $\sqrt{k(n)}\|p_n\|_{k(n)}^{1/k(n)} > \omega$ for all $n \geqslant N$. Then, for any $n \geqslant N$,

$$\inf_{m \geqslant 0} \sqrt{d(n,m)}\|p_{n,m}\|_{d(n,m)}^{1/d(n,m)} = \inf_{m \geqslant 0} \sqrt{k(n+m)}\|p_{n+m}\|_{k(n+m)}^{1/k(n+m)} \geqslant \omega,$$

hence (3) is satisfied. □

Remark 1. The growth conditions on (p_n) imply that $k(n)$ must satisfy the following: There is an $R > 0$ such that, for any $r > 0$, $R^n/r^{k(n)} \to \infty$ as $n \to \infty$. Hence, in particular, we cannot take $k(n) = n$, or that $k(n)$ grows even faster. However, conversely, for any sequence $(k(n)) \subseteq \mathbb{N}$ such that for some $R > 0$, $\lim_n R^n/r^{k(n)} = \infty$ for every $r > 0$, we can construct a sequence

(p_n) that satisfies the required growth conditions. We give an example below in the case of one variable.

Example 3. Let $1 \geqslant \beta > \alpha > 0$. Put $k(n) = [n^\alpha]$ and $p_n \equiv 2^{n^\beta} z^{k(n)}/k(n)!$, i.e. $p_n(D) = 2^{n^\beta} D^{k(n)}/k(n)!$. Then $\|p_n\|_{k(n)} = 2^{n^\beta}/\sqrt{k(n)!}$. It is easily checked that p_n satisfies the growth conditions in Corollary 2, so the elements

$$\Phi_n(D) = \sum_{m \geqslant 0} H_n p_{n+m}(D) = \sum_{m \geqslant 0} 2^{(n+m)^\beta} H_m D^{k(n+m)}/k(n+m)!,$$

$n = 0, 1, \ldots$ form a hypercyclic sequence $(\Phi_n(D))$.

We recall that every non-trivial convolution operator $\varphi(D)$ is hypercyclic. This means that the sequence $(\Phi_n(D))$, $\Phi_n(D) = \sum_m H_m \varphi^n(D)$ $(= \varphi(D)^n = \varphi^n(D))$, is hypercyclic. Thus, it is a natural question to ask for what sequences (φ_n), others than the constant ones $(\varphi_n = \varphi)$, the sequences $\Phi_n = (\varphi_{n,m} \equiv \varphi_m^n)_m$ of powers define a hypercyclic sequence $(\Phi_n(D))$.

Corollary 3 (*Powers*). *Let $\Phi = (p_n)$ be a sequence of k-homogeneous polynomials, where $k \geqslant 1$, such that $c \leqslant \|p_n\|_k \leqslant MR^n$ for some $c, M, R > 0$. Then $(\Phi_n(D))$, $\Phi_n \equiv (p_m^n)$, forms a (hereditarily) hypercyclic sequence in $\mathcal{O}(\mathbb{H})$.*

Proof. With $p_{n,m} \equiv p_m^n$, $d(n,m) = \deg p_{n,m} = kn$ and, by (1), $\|p_{n,m}\|_{d(n,m)} \geqslant c^n$. Theorem 1 implies that $(\Phi_n(D))$ is hereditarily hypercyclic for the full sequence (n). □

In our last application of Theorem 1, we consider sequences of the form $(\Phi_n = (p_{n,m} = p_n)_m)_n$, i.e. sequences $(p_n(D))$ of convolution operators $p_n(D)$. The following result complements Proposition 2:

Corollary 4 (*Convolution-sequences*). *Let (p_n) be a sequence of non-zero homogeneous polynomials p_n of degree $k(n)$, respectively. Then $(p_n(D))$ is hereditarily hypercyclic if there exists a sequence $(n_k) \subseteq \mathbb{N}$ such that $(k(n_k))_k$ is unbounded and*

$$\lim_{k\to\infty} \sqrt{k(n_k)} \|p_{n_k}\|_{k(n_k)}^{1/k(n_k)} = \infty. \tag{6}$$

Proof. Since $(k(n_k))$ is unbounded, we can extract a subsequence (m_k) of (n_k) such that $\lim_k k(m_k) = \infty$, and it is clear that $\lim_k \sqrt{k(m_k)} \|p_{m_k}\|_{k(m_k)}^{1/k(m_k)} = \infty$. Theorem 1 gives that $(p_n(D))$ is hereditarily hypercyclic for the sequence (m_k). □

Example 4. Any normalized sequence (p_n), i.e. $\|p_n\|_{\deg p_n} = 1$, of homogeneous polynomials, with unbounded degrees, defines thus a hypercyclic sequence $(p_n(D))$.

When $d = 1$ (note that, in this case, homogeneous polynomials are monomials and $\|cz^m\|_m = |c|\sqrt{m!}$), we can prove that the hypothesis of Corollary 4 is in fact necessary:

Theorem 2. *Let (p_n) be a sequence of non-zero monomials $p_n(z) = \lambda_n z^{k(n)}$ of degree $k(n)$, respectively. Then $(p_n(D))$ is hereditarily hypercyclic if and only if there is a sequence $(n_k) \subseteq \mathbb{N}$ such that $(k(n_k))_k$ is unbounded and $\lim_k \sqrt{k(n_k)} \|p_{n_k}\|_{k(n_k)}^{1/k(n_k)} = \infty$, i.e.,*

$$\lim_{k\to\infty} k(n_k) |\lambda_{n_k}|^{1/k(n_k)} = \infty. \tag{7}$$

Proof. First of all, that (7) is equivalent to $\lim_k \sqrt{k(n_k)}\|p_{n_k}\|_{k(n_k)}^{1/k(n_k)} = \infty$ follows by Stirling's formula, see the proof of Theorem 1.

Assume $(p_n(D))$ is hereditarily hypercyclic for some sequence (m_k). We prove that $(k(m_k))_k$ is unbounded. Assume not, i.e., assume $k(m_k) \leqslant m$ for all k. Let f be a hypercyclic vector for $(p_{m_k}(D))$. $\{p_{m_k}(D)f\}_k$ is contained in $\operatorname{Im} T$ where $T : \mathcal{P}_m \ni p \mapsto p(D)f \in \mathcal{H}$ and $\mathcal{P}_m$ denotes the (finite-dimensional) space of polynomials of degree at most m. Hence $\operatorname{Im} T$ is finite-dimensional which contradicts that f is hypercyclic. Thus there is a subsequence (l_k) of (m_k) such that $k(l_k) \to \infty$. In particular, $(p_{l_k}(D))$ is hypercyclic, so there exist a $g \in \mathcal{H}$ and a subsequence (n_k) of (l_k) such that $p_{n_k}(D)g = \lambda_{n_k} D^{k(n_k)} g \to 1$ in $\mathcal{H}$. This implies, in particular, that $(\lambda_{n_k} D^{k(n_k)} g(0))_{k \geqslant k_0}$ is bounded away from zero for some k_0, and we can assume $k_0 = 0$. We have that $k(n_k) \geqslant \varepsilon (k(n_k)!)^{1/k(n_k)}$ for all k for some $\varepsilon > 0$ and hence,

$$k(n_k)|\lambda_{n_k}|^{1/k(n_k)} \geqslant \varepsilon |\lambda_{n_k} D^{k(n_k)} g(0)|^{1/k(n_k)} (|D^{k(n_k)} g(0)|/k(n_k)!)^{-1/k(n_k)} \to \infty,$$

since $(|D^{k(n_k)} g(0)|/k(n_k)!)^{1/k(n_k)} \to 0$ as g is entire and $k(n_k) \to \infty$. Hence (7) holds and $(k(n_k))$ is unbounded.

The implication in the other direction is contained in Corollary 4. □

Part of Theorem 2.13 in [8] (see also [3, Theorem 4]) is essentially the special case $k(n) = n$ in our Theorem 2.

Corollary 4 (Theorem 2) provides us with hypercyclic sequences of convolution operators that do not satisfy any of the conditions (P) and (Q) in Proposition 2 (see also [8, Examples 1–3]):

Example 5. We obtain from Theorem 2 that the sequence $(p_n(D) = D^n/\sqrt{n!})$ is hypercyclic. Note that $p_n(a) = a^n/\sqrt{n!} \to 0$ for every $a \in \mathbb{C}$. Hence $(p_n(D))$ does not satisfy any of the conditions (P) and (Q) in Proposition 2.

Example 6. Any entire function $f = \sum f_n z^n$, that is not a polynomial, defines a hypercyclic sequence in the following way: Let (n_k) denote the strictly increasing sequence defined by $f_{n_k} \neq 0$. That is, n_1 is the first $n \in \mathbb{N}$ such that $f_n \neq 0$, n_2 the second such n and so on. Then we conclude from Theorem 2 that $(f_{n_k}^{-1} D^{n_k})_k$ is hypercyclic.

3. Existence of hypercyclic subspaces

Theorem 3. *Let $(\Phi_n = (p_{n,m})_m)_n$ be a sequence in $\mathcal{S}$ where $p_{n,m}$ are homogeneous polynomials. Assume there is a sequence $(n_k) \subseteq \mathbb{N}$, for which $(p_{n,m})$ satisfies the hypothesis of Theorem 1, and such that (the variety) $\cap_{k,m}\{a : p_{n_k,m}(a) = 0\}$ is infinite. Then $(\Phi_n(D))$ has a hypercyclic subspace.*

Proof. The proof of Theorem 1 shows that $(\Phi_n(D))$ satisfies the HC for the sequence (n_k), and we shall apply Proposition 5. If $\varphi \in \operatorname{Exp}$, we have that $e_a \equiv e^{\langle \cdot, a\rangle} \in \ker \varphi(D)$ ($\langle a, b\rangle \equiv \sum a_i b_i$) for any $a \in \mathbb{C}^d$ in the zeroset for φ. Indeed, $\varphi(D)e_a = \varphi(a)e_a = 0$. Further, the elements e_a, $a \in \mathbb{C}^d$, form a linearly independent set. Hence, the hypothesis implies that $E \equiv \cap_k \ker \Phi_{n_k}(D)$ is infinite-dimensional (and, of course, closed). Evidently, $\Phi_{n_k}(D) \to 0$ pointwise on E and hence the theorem. □

We leave it to the reader to apply Theorem 3 for the (special type of) sequences of operators in Corollaries 1, 2 and 3. (In fact, the sequences (operators) in Corollary 1 will be studied in detail in the next section.) However, we formulate its consequence when we apply it on the type of sequences in Corollary 4:

Corollary 5. *Let (p_n) be a sequence of non-zero homogeneous polynomials p_n of degree $k(n)$, respectively. Assume there is a sequence $(n_k) \subseteq \mathbb{N}$ such that*: (i) $(k(n_k))_k$ *is unbounded* (ii) (6) *holds and* (iii) $\cap_k\{a : p_{n_k}(a) = 0\}$ *is infinite. Then $(p_n(D))$ has a hypercyclic subspace.*

Proof. Properties (i) and (ii) imply that $(p_{n,m} \equiv p_n)$ satisfies the hypothesis of Theorem 1 for some subsequence (m_k) of (n_k) (see the proof of Corollary 4), and (iii) implies that $\cap_k\{a : p_{m_k,m}(a) = p_{m_k}(a) = 0\}$ is infinite and hence the corollary by Theorem 3. □

4. The set of hypercyclic subspaces

It is convenient to denote by $\mathcal{H}(T)$ the set (possibly empty) of hypercyclic subspaces for an operator $T \in \mathcal{L}(\mathcal{X})$, and we call $\mathcal{H}(T)$ the hypercyclic spectrum of T. By a result of Ansari, see [1], T and any positive power of T have the same set of hypercyclic vectors, consequently, $\mathcal{H}(T) = \mathcal{H}(T^n)$ for any positive $n \in \mathbb{N}$. We recall from the Introduction that T lacks non-trivial closed invariant subsets if and only if $\mathcal{X} \in \mathcal{H}(T)$. Further, if $H \in \mathcal{H}(T)$ is invariant under T, H must be the entire space $\mathcal{X}$. Observations like these, and Theorem 5, motivate us to study the structure of the hypercyclic spectrum for hypercyclic operators. More specifically, an application of Theorem 3, and the proof of Corollary 1, give that $\mathcal{H}(T) \neq \emptyset$ for any operator $T = \Phi(D) = \sum H_n p_n(D)$ where:

(i) $\{p_n\}_{n\geqslant 0} \subseteq \mathcal{H}_m \backslash \{0\}$ for some $m \geqslant 1$;
(ii) $c \leqslant \|p_n\|_m \leqslant MR^n/\sqrt{n!}$ for some $M, R, c > 0$ and all $n \geqslant 0$ (m is as in (i));
(iii) $\cap_{n\geqslant 0}\{a : p_n(a) = 0\}$ is infinite.

The primary objective in this section is to investigate the structure of $\mathcal{H}(T)$ for such operators. It is evident that $\mathcal{H}$ does not belong to the hypercyclic spectrum for any such operator $\Phi(D)$, since $\Phi(D)1 = 0$.

We start with the following general:

Proposition 8. *Let $T \in \mathcal{L}(\mathcal{F})$ be a surjective operator on a Fréchet space $\mathcal{F}$ and let $H \in \mathcal{H}(T)$, $H \neq \mathcal{F}$. Then $H(m) \equiv (T^m)^{-1}H$, $m = 0, 1, \ldots$ are different elements of $\mathcal{H}(T)$. More specifically, $H(n)\backslash H(m) \neq \emptyset$ if $m > n \geqslant 0$. (Note that $H(0) = H$.)*

Proof. Since H is closed, any $H(m)$ is closed. By the surjectivity of T^m, $T^m H(m) = H$ and from this we conclude that $H(m)$ is infinite-dimensional and that the non-zero vectors of $H(m)$ are hypercyclic. Thus, $H(m) \in \mathcal{H}(T)$ for every $m \geqslant 0$. Next, assume $H(n) \subseteq H(m)$ for some $m > n \geqslant 0$. Then $H = T^m H(m) \supseteq T^{m-n}T^n H(n) = T^{m-n}H$. Since $\mathcal{H}(T) = \mathcal{H}(T^{m-n})$, $H \in \mathcal{H}(T^{m-n})$ so H must be the entire set $\mathcal{F}$, which is a contradiction. □

Proposition 9. *Any operator $\Phi(D) = \sum H_n p_n(D)$ where (p_n) satisfies* (i) *and* (ii) *is surjective.*

Proof. We put $\mathcal{H}$ and Exp into duality in the following way. For each n we define the bilinear form $\langle\cdot,\cdot\rangle_n$ on $\mathcal{H}_n$ by $\langle p, q\rangle_n \equiv (p, \bar{q})_n$. We obtain then a bilinear form $\langle\cdot,\cdot\rangle$ on $\mathcal{H} \times \text{Exp}$ by $\langle f, \varphi\rangle \equiv \sum \langle H_n f, H_n \varphi\rangle_n$. This duality is sometimes referred to as the Martineau-duality, and it is well-known that Exp $\simeq \mathcal{H}'$ in this way (see [32, p. 523] for further remarks). Hence, since $\mathcal{H}$ is Fréchet, is suffices to prove that the transpose ${}^t\Phi(D)$ is one-to-one and has weakly closed range (i.e. $\ker \Phi(D)^\perp \subseteq \mathrm{Im}\,{}^t\Phi(D)$). Now, by noting that ${}^tp_n(D) = p_n$ and ${}^tH_n = H_n$, we deduce that ${}^t\Phi(D) = \sum p_n H_n$. Hence it is evident that ${}^t\Phi(D)$ is one-to-one. Assume next that $\varphi \in \ker \Phi(D)^\perp$. That $f = \sum f_n \in \ker \Phi(D)$ means $0 = H_n p_n(D) f = p_n(D) H_{n+m} f$ for all n. So for any $f \in \mathcal{H}_{n+m} \in \ker p_n(D)$, $f \in \ker \Phi(D)$ and so $0 = \langle f, \varphi \rangle = \langle f, H_{n+m}\varphi \rangle_{n+m}$. This shows that every $H_{n+m}\varphi \in p_n \mathcal{H}_n$, for $p_n : \mathcal{H}_n \to \mathcal{H}_{n+m}$ is the transpose of $\varphi_n(D) : \mathcal{H}_{n+m} \to \mathcal{H}_n$ and so $\ker p_n(D)^\perp = \mathrm{Im}\, p_n = p_n \mathcal{H}_n$. Hence, for each n there exists a $\psi_n \in \mathcal{H}_n$ such that $H_{n+m}\varphi = \psi_n p_n$, and thus, $\varphi = \sum \psi_n p_n$. It remains thus only to prove that $\sum \psi_n \in \mathrm{Exp}$, i.e., that $\|\psi_n\|_n \leqslant MR^n/\sqrt{n!}$ for some $M, R > 0$. But since $\varphi \in \mathrm{Exp}$, $\|H_n\varphi\|_n \leqslant MR^n/\sqrt{n!}$ for some $M, R > 0$ and (1) gives

$$c\|\psi_n\|_n \leqslant \|p_n\|_m \|\psi_n\|_n \leqslant \|H_{n+m}\varphi\|_{n+m} \leqslant M \frac{R^{n+m}}{\sqrt{(n+m)!}} \leqslant M \frac{R^n R^m}{\sqrt{n!}\sqrt{m!}}, \tag{8}$$

so $\sum \psi_n \in \mathrm{Exp}$ and the proof is complete. □

Remark 2. More generally, we see from (8) and the arguments in the proof that any operator $\Phi(D) = \sum H_n p_n(D)$ satisfying (i) and the following condition (ii)′ (weaker than (ii)): $cr^n \leqslant \|p_n\|_m \leqslant MR^n/\sqrt{n!}$ for some $M, R, r, c > 0$, is surjective on $\mathcal{H}$.

Proposition 10. *Let $T \in \mathcal{L}(\mathcal{F})$ be a hypercyclic operator on a Fréchet space $\mathcal{F}$, and assume that T has a continuous right inverse S. If H is a hypercyclic subspace for T with $H \neq \mathcal{F}$, $S^m H$, $m \geqslant 0$, form different elements of $\mathcal{H}(T)$, in fact, $S^n H \setminus S^m H \neq \emptyset$ when $m > n \geqslant 0$.*

Proof. We note that any continuous right inverse S is one-to-one and has closed range. Indeed, ST forms a projector onto $\mathrm{Im}\, S$ so the image $\mathrm{Im}\, S$ is closed. That S must be injective is obvious. Accordingly, the image SH is closed and infinite-dimensional since H is. If x is hypercyclic for T then so is $\tilde{x} \equiv Sx$, since $\{T^n \tilde{x}\}_n \supseteq \{T^n x\}_n$. Thus SH, and therefore any $S^n H$, forms a hypercyclic subspace.

Next, assume first $H \subseteq SH$. Then $TH \subseteq TSH = H$, i.e. T maps H invariantly, so H must be the entire space $\mathcal{F}$ which is not the case. Hence $H \setminus SH \neq \emptyset$.

Assume now that $S^n H \subseteq S^m H$ for some $m > n \geqslant 0$. Then $H = T^n S^n H \subseteq T^n S^m H = S^{m-n} H$. But $H \in \mathcal{H}(T) = \mathcal{H}(T^{m-n})$ and S^{m-n} is a right inverse to T^{m-n}, so, from what we just have proved, $H \setminus S^{m-n} H \neq \emptyset$ and we have a contradiction. □

Proposition 11. *Let $T \in \mathcal{L}(\mathcal{F})$ be an operator on Fréchet space $\mathcal{F}$ that admits a continuous norm, and assume T satisfies the HC. Assume $T - \lambda$ has infinite-dimensional kernel and a continuous right inverse $S \in \mathcal{L}(\mathcal{F})$ for some $|\lambda| < 1$. Then T admits a complemented hypercyclic subspace $H \simeq \ker(T - \lambda)$.*

Proof. We apply Proposition 6, and note first that since $T - \lambda$ has a right inverse S, $E \equiv \ker(T - \lambda)$ is complemented in $\mathcal{F}$ with $\mathcal{F} = E \oplus \mathrm{Im}\, S$. Indeed, $\Pi \equiv S(T - \lambda)$ is a projector onto $\mathrm{Im}\, S$ with kernel $\ker(T - \lambda)$. Since $|\lambda| < 1$, $T^{n_k} \to 0$ pointwise on E for any strictly increasing sequence (n_k), and in particular for any strictly increasing sequence (n_k) for which T satisfies the HC. □

Proposition 12. *Any operator $\Phi(D) = \sum H_n p_n(D)$ where (p_n) satisfies* (i) *and* (ii) *has a continuous right inverse. Such an inverse S is given by*

$$S = \sum_{n \geqslant 0} \bar{p}_n (p_n(D)\bar{p}_n)^{-1} H_n. \tag{9}$$

Proof. See the proof of Theorem 1. □

We now piece all things (Propositions 8–12) together, and obtain the following main result in this section:

Theorem 4. *Assume* (p_n) *satisfies* (i) *through* (iii) *and consider the operator* $\Phi(D) = \sum H_n p_n(D)$. *Then* $\mathcal{H}(\Phi(D)) \neq \emptyset$ *but* $\mathcal{H} \notin \mathcal{H}(\Phi(D))$, *in fact*, $\mathcal{H}(\Phi(D))$ *contains a complemented* $H \simeq \ker \Phi(D)$. *Further, for any* $H \in \mathcal{H}(\Phi(D))$ *and right inverse* S *to* $\Phi(D)$, *e.g.* (9), *we have*:

1. $\{H(m) \equiv (\Phi(D)^m)^{-1} H\}_{m \geqslant 0} \subseteq \mathcal{H}(\Phi(D))$ *and* $H(n) \setminus H(m) \neq \emptyset$ *if* $m > n \geqslant 0$;
2. $\{S^m H\}_{m \geqslant 0} \subseteq \mathcal{H}(\Phi(D))$ *and* $S^n H \setminus S^m H \neq \emptyset$ *if* $m > n \geqslant 0$.

Moreover, $S^m H \neq H(n)$ *for all* n *and* m *unless* $n = m = 0$.

Proof. It remains only to prove the last statement. Put $T \equiv \Phi(D)$. Assume first that $S^m H = H(n)$ where $n > m$. Then, by the surjectivity of T^n, $H = T^n H(n) = T^n S^m H = T^{n-m} H$ which is a contradiction since $H \in \mathcal{H}(T^{n-m})$ and $H \neq \mathcal{H}$. Next, assume $S^m H = H(n)$ where $m \geqslant n$ and not both are equal to 0. When $n = 0$, and thus $m > 0$, this is not possible by 2, so assume $m \geqslant n > 0$. Let $f \in S^m H = H(n)$. T^n is not injective, so we can find a non-zero $g \in \ker T^n \subseteq \ker T^m$. Then $h \equiv f + g \in H(n) \setminus S^m H$—a contradiction. Indeed, it is evident that $h \in H(n)$, and since S^m is a continuous right inverse to T^m, we have $\mathcal{H} = \ker T^m \oplus \operatorname{Im} S^m$, from which we conclude that $h \notin \operatorname{Im} S^m$. □

We conclude this section with a result that applies to arbitrary sequences. Let $\mathcal{P}_m$ denote the vector space of polynomials of degree at most m. If K is a continuous linear operator on $\mathcal{H}$ with $\operatorname{Im} K \subseteq \mathcal{P}_m$, K is compact and from operator theory we know that $I - K$ forms an operator with closed range and finite-dimensional kernel. (I denotes here the identity operator.) These observations give:

Proposition 13. *Let* $\Phi_n = (p_{n,m})_m$, $n \geqslant 0$, *be sequences of homogeneous polynomials* $p_{n,m}$ *that satisfy the hypothesis of Theorem* 3 *for the full sequence* (n). *Then* $(I - K)H$ *forms a hypercyclic subspace for* $(\Phi_n(D))$ *for any hypercyclic subspace* H *for* $(\Phi_n(D))$ *and any* $K \in \mathcal{L}(\mathcal{H})$ *with* $\operatorname{Im} K \subseteq \mathcal{P}_m$ *for some* m.

Proof. Since $I - K$ has closed range and finite-dimensional kernel, $(I - K)H$ forms an infinite-dimensional closed subspace. It suffices thus to prove that $(I - K)f$ is hypercyclic for any hypercyclic f. This follows by the simple observation that $\Phi_n(D)f = \Phi_n(D)(I - K)f$ for all n sufficiently large. Indeed, since $\lim_n \deg p_{n,k} = \infty$, $\mathcal{P}_m \subseteq \ker \Phi_n(D)$ when n is large. □

Example 7. Let $K = K_\tau$ be the *Taylor projector of order* m, that maps any $f = \sum f_n \in \mathcal{H}$ onto its Taylor polynomial $\sum_{n \leqslant m} f_n$ of order m about the origin. Then $\operatorname{Im} K \subseteq \mathcal{P}_m$ so Proposition 13 applies and we note that $I - K = \sum_{n > m} H_n$. (Note that $K_\tau \in \mathcal{O}(\mathbb{H})$, and the projectors in $\mathcal{O}(\mathbb{H})$ onto the spaces $\mathcal{P}_m$, for which Proposition 13 thus applies, are described in [26].)

5. Consequences and remarks

We recall the result of Bonet et al. [13] saying that if $\mathbb{T} \subseteq \mathcal{L}(\mathcal{F})$ satisfies the HC, where $\mathcal{F}$ is a separable Fréchet space admitting a continuous norm, then we can find a compact operator

$K \in \mathcal{L}(\mathcal{F})$ that maps non-zero vectors onto hypercyclic vectors for $\mathbb{T}$. We now prove that another sufficient condition for the existence of such an operator K, is the existence of a hypercyclic subspace H for $\mathbb{T}$, and the proof, that also can be found in [31], shows how we can construct K out from H:

Theorem 5. *Let $\mathbb{T} = (T_n)$ be a sequence of operators $T_n \in \mathcal{L}(\mathcal{F})$ on a separable Fréchet space $\mathcal{F}$ admitting a continuous norm. Assume $\mathbb{T}$ has a hypercyclic subspace H. Then there is a compact one-to-one operator $K \in \mathcal{L}(\mathcal{F})$ with $\mathrm{Im}\, K \subseteq H$, and thus, K maps non-zero vectors onto hypercyclic vectors for $\mathbb{T}$.*

Proof. The hypothesis that $\mathcal{F}$ admits a continuous norm implies we can find a countable equicontinuous $\sigma(\mathcal{F}', \mathcal{F})$-total subset $\{f_n\}$ of $\mathcal{F}'$, see [13, p. 606]. Next, since H is closed and infinite-dimensional, H forms an infinite-dimensional Fréchet space and hence contains a basic sequence (e_n), see [12,19]. (Mazur's classical proof of that every infinite-dimensional Banach space contains a basic sequence is exposed in [21, Theorem 1.a.5].) We can assume (e_n) is bounded, since otherwise we multiply by suitable scalars. Choose a sequence $(\lambda_n) \in \ell_1$ such that $\lambda_n \neq 0$. Then $K \equiv \sum_{n \geqslant 0} \lambda_n \langle \cdot, f_n \rangle e_n$ is a nuclear, and hence a compact, operator. Since (f_n) separates points in $\mathcal{F}$ and (e_n) is basic, K is one-to-one, and $\mathrm{Im}\, K \subseteq H$ as H is closed and $(e_n) \subseteq H$. $\square$

Remark 3. Assume $\mathcal{F}$ is a separable Fréchet space with a continuous norm. In view of Theorem 5 and its preceding paragraph, one may wonder whether every sequence $\mathbb{T} \subseteq \mathcal{L}(\mathcal{F})$ with a hypercyclic subspace satisfies the HC (recall that the converse is not true by [24, Theorem 3.4]). Let us show with an example that this is, in general, false. For this, take for instance the space $\mathcal{F} \equiv \ell_1$ of summable sequences $x = (x_1, x_2, \ldots)$ (i.e. $\sum |x_n| < \infty$), and choose any hypercyclic operator $T \in \mathcal{L}(\mathcal{F})$ supporting a hypercyclic subspace H (see [20, Corollary 2.2]). Inspired by the proof of Theorem 2.3 in [5], we define, for each $n \geqslant 1$ and each $x \in \ell_1$,

$$T_n x \equiv T^n \tilde{x} + (1 + \|T\|)^n \hat{x},$$

where $\tilde{x} \equiv (x_2, x_3, \ldots)$ (= the forward-shifted vector of x) and $\hat{x} \equiv (x_1, 0, 0, \ldots)$ (= the 1-projected vector of x). It is evident that every $T_n \in \mathcal{L}(\mathcal{F})$, that is, $\mathbb{T} \equiv (T_n)$ is a sequence in $\mathcal{L}(\mathcal{F})$. Observe that if $x_1 \neq 0$ then

$$\|T_n\| \geqslant (1 + \|T\|)^n |x_1| - \|T\|^n \|\tilde{x}\| \to \infty \quad (n \to \infty),$$

so $\mathrm{hc}(\mathbb{T}) \subseteq \{x \in \ell_1 : x_1 = 0\}$. Therefore, the set of hypercyclic vectors for $\mathbb{T}$, $\mathrm{hc}(\mathbb{T})$, is not dense in $\mathcal{F} = \ell_1$, hence $\mathbb{T}$ does not satisfies the HC. Now, let $H_1 \equiv \{(0, x_1, x_2, \ldots) : (x_1, x_2, \ldots) \in H\}$. It is evident that H_1 is a closed infinite-dimensional subspace of ℓ_1. Finally, if $y = (0, x_1, x_2, \ldots) \in H_1 \backslash \{0\}$ then $\tilde{y} \equiv (x_1, x_2, \ldots) \in H \backslash \{0\}$, whence $\{T^n \tilde{y} : n \geqslant 1\}$ is dense in ℓ_1. But $T_n y = T^n \tilde{y} + 0$ $(n \geqslant 1)$, so $\{T_n y : n \geqslant 1\}$ is also dense in ℓ_1, that is, the vector y is hypercyclic for $\mathbb{T}$. Consequently, H_1 is a hypercyclic subspace for $\mathbb{T}$.

Our goal is now to show how, given a hypercyclic sequence $\mathbb{T} \subseteq \mathcal{L}(\mathcal{H})$, we can obtain a *holomorphic* mapping $\mathbb{C}^d \to \mathcal{H}(\mathbb{C}^d)$ whose values are hypercyclic vectors for $\mathbb{T}$. This will be provided via *linear* mappings K such that in Theorem 5 and by virtue of a kernel-theorem for $\mathcal{L}(\mathcal{H})$. In particular, this result will apply to the obtained hypercyclic sequences in this note (and of course to the hypercyclic sequences in Proposition 2, especially to non-trivial convolution operators, see the remark at the end).

We start with a definition:

Definition 3. By $\mathfrak{S}$ we denotes the set of entire mappings $P = P(z, \xi)$ in $2d$ variables $(z, \xi) \in \mathbb{C}^d \times \mathbb{C}^d$ with the following property: For every $r \geqslant 0$ there are constants $R, M > 0$ such that $\sup_{\|z\| \leqslant r} |P(z, \xi)| \leqslant M e^{R\|\xi\|}$ for all $\xi \in \mathbb{C}^d$.

Note that if $P \in \mathfrak{S}$ then $P(\cdot, \xi) \in \mathcal{H}$ and $P(z, \cdot) \in \mathrm{Exp}$ for any fixed ξ and z, respectively. For fixed $a \in \mathbb{C}^d$ we put $e_a \equiv e^{\langle \cdot, a \rangle} \in \mathcal{H}$ ($\langle z, \xi \rangle \equiv \sum z_i \xi_i$). Let us state now the following kernel theorem for $\mathcal{L}(\mathcal{H})$ (see [28, Theorem 1] for a proof, and cf. Proposition 1):

Proposition 14 (*Kernel-theorem*). *The map $T \mapsto P(z, \xi) \equiv e^{-\langle z, \xi \rangle} T e_\xi(z)$ defines a bijection between $\mathcal{L}(\mathcal{H})$ and $\mathfrak{S}$. We write $T = P(\cdot, D)$ and have*

$$Tf(z) = \sum_{(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d} P_{\alpha, \beta} z^\alpha D^\beta f(z), \tag{10}$$

where $P(z, \xi) = \sum_{\alpha, \beta} P_{\alpha, \beta} z^\alpha \xi^\beta$, with convergence in $\mathcal{H}$.

The unique function $P \in \mathfrak{S}$ is called the symbol for $T = P(\cdot, D)$. Note that the symbol P for a $\varphi(D) \in \mathcal{C}$ is $P(z, \xi) = \varphi(\xi)$.

Theorem 6. *Let $\mathbb{T} = (T_n)$ be a sequence in $\mathcal{L}(\mathcal{H})$ and assume that $\mathbb{T}$ satisfies the HC or that $\mathbb{T}$ has a hypercyclic subspace. Then there is an $h \in \mathfrak{S}$ such that $h(\cdot, \xi) \in \mathcal{H}(\mathbb{C}^d)$ is hypercyclic for $\mathbb{T}$ for every fixed $\xi \in \mathbb{C}^d$. Such an h is given by $h(z, \xi) \equiv Se_\xi(z)$ where $S \in \mathcal{L}(\mathcal{H})$ is any operator that maps non-zero vectors onto hypercyclic vectors for $\mathbb{T}$ (exists by our hypothesis).*

Proof. Put $h(z, \xi) \equiv Se_\xi(z)$. Then $h \in \mathfrak{S}$ by Proposition 14 and, since $e_\xi \neq 0$, $h(\cdot, \xi)$ is hypercyclic for any fixed ξ. □

Corollary 6. *For any sequence $(\Phi_n = (p_{n,m})_m)_n$ in $\mathcal{S}$ satisfying the hypotheses of Theorem* 1, *there is an $h \in \mathfrak{S}$ such that for any $f \in \mathcal{H}$, $\xi \in \mathbb{C}^d$, $\varepsilon > 0$ and $r > 0$*:

$$\sup_{\|z\| \leqslant r} |[\Phi_n(D) h(\cdot, \xi) - f](z)| \leqslant \varepsilon$$

for some n.

6. A closing remark

Many of the arguments in the previous sections apply to (non-trivial) convolution operators. First of all, the proof of [16, Theorem 5.2] shows that any non-trivial convolution operator on $\mathcal{H}$ satisfies the HC for the full sequence. Next we conclude that for any $\varphi \in \mathrm{Exp}$, that is not a constant mapping and, in the case $d = 1$, not a polynomial, $\mathcal{H}(\varphi(D)) \neq \emptyset$. In fact, $\mathcal{H}(\varphi(D))$ contains a complemented subspace. Indeed, we only have to note that for any such φ, the zeroset $\{a : \varphi(a) - \lambda = 0\}$ for $\varphi - \lambda$ is infinite for some λ in the open unit disc (details on this are given in [30]). Therefore, see the proof of Theorem 3, $\ker(\varphi(D) - \lambda)$ is infinite-dimensional. Moreover, by a result of Meise and Taylor [23], any non-zero convolution operator (and hence $\varphi(D) - \lambda$) has a continuous right inverse, and so $\varphi(D)$ has a complemented hypercyclic subspace by Proposition 11. (It is not known whether any ordinary differential operator $p(D) \in \mathcal{L}(\mathcal{H}(\mathbb{C}))$

has a hypercyclic subspace, in particular—does D have non-empty hypercyclic spectrum?) The result of Meise and Taylor implies also that Proposition 10 can be applied. (However, in this case we have no general formula for the right inverses.) Moreover, Malgrange's classical theorem [22, Theorem 5, p. 322] states that any non-zero convolution operator on $\mathcal{H}$ is surjective, hence Proposition 8 applies.

Finally, we conclude that for any non-trivial $\varphi(D) \in \mathcal{C}$ (d is arbitrary), Theorem 6 applies ($T_n = \varphi(D)^n = \varphi^n(D)$).

Acknowledgements

I would like to thank the referees for the mathematical/stylistic/bibliographic suggestions improving the original manuscript. In particular, the example in Remark 3 is due to one of the referees, and I am grateful to him/her for letting me complement (and improve) the paper by this nice example.

References

[1] S.I. Ansari, Hypercyclic and cyclic vectors, J. Funct. Anal. 128 (1995) 374–383.
[2] R. Aron, D. Markose, On universal functions, J. Korean Math. Soc. 41 (1) (2004) 65–76.
[3] L. Bernal-González, Derivative and antiderivative operators and the size of complex domains, Ann. Polon. Math. 94 (1994) 267–274.
[4] L. Bernal-González, Hypercyclic sequences of differential and antidifferential operators, J. Approx. Theory 96 (1999) 323–337.
[5] L. Bernal-González, Norms of hypercyclic sequences, Math. Pann. 15 (2004) 221–230.
[6] L. Bernal-González, Hypercyclic subspaces in Fréchet spaces, Proc. Amer. Math. Soc., to appear.
[7] L. Bernal-González, A. Montes-Rodríguez, Non-finite dimensional closed vector spaces of universal functions for composition operators, J. Approx. Theory 82 (1995) 375–391.
[8] L. Bernal-González, J.A. Prado-Tendero, Sequences of differential operators: exponentials, hypercyclicity and equicontinuity, Ann. Polon. Math. 77 (2001) 169–187.
[9] J. Bès, Invariant manifolds of hypercyclic vectors for the real scalar case, Proc. Amer. Math. Soc. 127 (1999) 1801–1804.
[10] J. Bès, A. Conejero, Hypercyclic subspaces in omega, J. Math. Anal. Appl., to appear.
[11] J. Bès, A. Peris, Hereditarily hypercyclic operators, J. Funct. Anal. 167 (1999) 94–112.
[12] C. Bessaga, A. Pelczyński, Wlasności Baz w Przestrzeniach Typu B, Prace. Mat. III (1959) 123–142.
[13] J. Bonet, F. Martínez, A. Peris, Universal and chaotic multipliers on spaces of operators, J. Math. Anal. Appl. 297 (2004) 599–611.
[14] P.S. Bourdon, Invariant manifolds of hypercyclic vectors, Proc. Amer. Math. Soc. 118 (1993) 845–847.
[15] K.C. Chan, R.D. Taylor, Hypercyclic subspaces of a Banach space, Integral Equations Operator Theory 41 (2001) 381–388.
[16] G. Godefroy, J. Shapiro, Operators with dense, cyclic vector manifolds, J. Funct. Anal. 98 (1991) 229–296.
[17] M. González, F. León-Saavedra, A. Montes-Rodríguez, Semi-Fredholm theory: hypercyclic and supercyclic subspaces, Proc. London Math. Soc. 81 (3) (2000) 169–189.
[18] K.-G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. (N.S.) 36 (3) (1999) 345–381.
[19] P.K. Kamthan, M. Gupta, Theory of Bases and Cones, Pitman, Boston, 1985.
[20] F. León-Saavedra, A. Montes-Rodríguez, Linear structure of hypercyclic vectors, J. Funct. Anal. 148 (1997) 524–545.
[21] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I, Springer, Heidelberg, 1977.
[22] B. Malgrange, Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, Ann. Inst. Fourier 6 (1955) 271–354.
[23] R. Meise, B.A. Taylor, Each non-zero convolution operator on the entire functions admits a continuous linear right inverse, Math. Z. 97 (1988) 139–152.
[24] A. Montes-Rodríguez, Banach spaces of hypercyclic vectors, Michigan Math. J. 43 (1996) 419–436.

[26] H. Petersson, The PDE-preserving operators on nuclearly entire functions of bounded type, Acta Math. Hungar. 100 (1–2) (2003) 69–81.
[27] H. Petersson, Rings of PDE-preserving operators on nuclearly entire functions, Studia Math. 163 (3) (2004) 217–229.
[28] H. Petersson, PDE-preserving properties, J. Korean Math. Soc. 42 (3) (2005) 573–597.
[29] H. Petersson, Supercyclic and hypercyclic non-convolution operators, J. Operator Theory, to appear.
[30] H. Petersson, Hypercyclic and subspaces for Fréchet space operators, J. Math. Anal. Appl., to appear.
[31] H. Petersson, Complemented hypercyclic subspaces, Houston J. Math., to appear.
[32] H. Shapiro, An algebraic theorem of E. Fischer, and the holomorphic Goursat problem, Bull. London Math. Soc. 21 (1989) 513–537.